Low complexity method for signal subspace fitting

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A low complexity method for signal subspace fitting is proposed. The novel signal subspace used in the method is spanned by the pre-filters of the multi-stage Wiener filter. Studies and simulations have shown that, when the incident signals are coherent, the new method achieves comparable results with the weighted subspace fitting estimator but requires much lower computational cost.

Introduction: It is shown in [1] that the weighted subspace fitting (WSF) method, a minimising technique, always outperforms the deterministic maximum likelihood (DML) estimator for the problem of the direction-of-arrival (DOA) estimation. However, the WSF estimator is still subject to high computational cost, partially attributable to the estimation of the signal subspace. In this Letter, the relation between the multi-stage Wiener filter (MSWF) [2] and the signal subspace is found, therefore leading to a low complexity method for signal subspace fitting (SSF). The new method does not require an estimate of the covariance matrix or its eigen-decomposition, thereby implying very low computational complexity. Moreover, since the new signal subspace can be spanned only by D matched filters of the MSWF, its dimension may be much less than the number of signals. Simulations will show that, when the incident signals are coherent, the novel estimator yields approximately consistent results with the WSF method.

Problem formulation: Let us consider a uniform linear array (ULA) of M isotropic sensors that receive $P(P<M)$ narrowband signals coming from directions $\{\theta_1, \theta_2, \ldots \theta_P\}$. The data, which are corrupted by additive noise, received by the array at the kth snapshot can be written as

$$\begin{aligned}\mathbf{x}(k) &= [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \ldots, \mathbf{a}(\theta_P)]\mathbf{s}(k) + \mathbf{n}(k)\\ &= \mathbf{A}(\theta)\mathbf{s}(k) + \mathbf{n}(k) \quad k = 0, 1, \ldots, N-1\end{aligned} \tag{1}$$

where $\mathbf{s}(k) \in C^{P\times 1}$, $\mathbf{n}(k) \in C^{M\times 1}$, $\mathbf{A}(\theta)$ $C^{M\times P}$ are the signal vector, the noise vector and the $M\times P$ steering matrix, respectively. N denotes the number of snapshots. The background noise uncorrelated with the signals is assumed to be a stationary Gaussian white random process, which is also spatially white and circularly symmetric. Therefore, the covariance matrix can be expressed as

$$\mathbf{R_x} = E[\mathbf{x}(k)\mathbf{x}^H(k)] = \mathbf{A}\mathbf{R_s}\mathbf{A}^H + \sigma_\mathbf{n}^2\mathbf{I}_{M\times M} \tag{2}$$

where $\mathbf{R_s}$ and $\sigma_\mathbf{n}^2$ are the signal covariance matrix and the noise variance, respectively. Computing the eigenvectors associated with the covariance matrix $\mathbf{R_x}$ yields

$$\mathbf{R_x} = \mathbf{V_s}\Lambda_\mathbf{s}\mathbf{V_s}^H + \sigma_\mathbf{n}^2\mathbf{V_n}\mathbf{V_n}^H \tag{3}$$

The number of the columns of $\mathbf{V_s}$ is generally equal to the rank P' of $\mathbf{R_s}$. Thus the columns of $\mathbf{V_s}$ span the P'-dimensional subspace of $\mathbf{A}(\theta)$. Combining (2) and (3) results in

$$\mathbf{V_s} = \mathbf{A}(\theta)\mathbf{T} \tag{4}$$

where $\mathbf{T} \in C^{P\times P'}$ is the full-rank matrix. Equation (4) forms a basis for the WSF method.

Low complexity method for signal subspace fitting: The MSWF recently presented by Goldstein *et al.* [2] can efficiently solve the *Wiener-Hopf* equation $\mathbf{R_x w} = \mathbf{r}_{\mathbf{x}d}$ in the minimum mean square error (MMSE) sense. The MSWF algorithm [3] based on the data-level lattice structure is shown as follows:

Initialisation: $d_0(k)$ and $\mathbf{x}_0(k) = \mathbf{x}(k)$.
Forward recursion: For $i =$ 1; 2, ..., M:

$$\begin{aligned}\mathbf{h}_i &= E[\mathbf{x}(k)_{i-1}d_{i-1}^*(k)]/\|E[\mathbf{x}(k)_{i-1}d_{i-1}^*(k)]\|_2;\\ d_i(k) &= \mathbf{h}_i^H\mathbf{x}_{i-1}(\mathbf{k});\\ \mathbf{x}_i(k) &= \mathbf{x}_{i-1}(k) - \mathbf{h}_i d_i(k).\end{aligned}$$

Backward recursion: For $i = M, M-1, \ldots, 1$ with $\varepsilon_M(k) = d_M(k)$:

$$\begin{aligned}w_i &= E[d_{i-1}(k)\varepsilon_i^*(k)]/E[|\varepsilon_i(k)|^2];\\ \varepsilon_{i-1}(k) &= d_{i-1}(k) - w_i^*\varepsilon_i(k).\end{aligned}$$

In the algorithm above, d_0 (k) is the reference signal that can be obtained from the spreading codes of users or the training sequences.

The pre-filter matrix $\mathbf{T}_D = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_D]$ $(D \le P)$ is acquired by truncating the MSWF at the Dth stage to reduce complexity. Notice that all the matched filters $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_D$ are orthogonal. It follows from [4] that the rank D MSWF is equivalent to solving the *Wiener-Hopf* equation in the D-dimensional *Krylov* subspace $K^{(D)} = span(\mathbf{r}_{\mathbf{x}_0 d_0};$ $\mathbf{R}_{\mathbf{x}_0}\mathbf{r}_{\mathbf{x}_0 d_0}, \ldots, \mathbf{R}_{\mathbf{x}_0}^{(D-1)}\mathbf{r}_{\mathbf{x}_0 d_0})$, i.e. $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_D$ form an orthogonal basis for the $K^{(D)}$. It readily follows that there exists a full-rank matrix $\mathbf{K} \in C^{D\times D}$ such that

$$\mathbf{T}_D = [\mathbf{r}_{\mathbf{x}_0 d_0}\ \mathbf{R}_{\mathbf{x}_0}\mathbf{r}_{\mathbf{x}_0 d_0}\ \cdots\ \mathbf{R}_{\mathbf{x}_0}^{(D-1)}\mathbf{r}_{\mathbf{x}_0 d_0}]\mathbf{K} \tag{5}$$

According to (3) and with $\mathbf{V_s}^H\mathbf{V_s} = \mathbf{I}_{P'\times P'}$ and $\mathbf{V_n}^H\mathbf{V_n} = \mathbf{I}_{(M-P')\times(M-P')}$ in mind, we have

$$\mathbf{R}_{\mathbf{x}_0}^{(d)} = \mathbf{V_s}\Lambda_\mathbf{s}^{(d)}\mathbf{V_s}^H + \sigma_\mathbf{n}^{2d}\mathbf{V_n}\mathbf{V_n}^H \quad d = 1, 2, \ldots, D-1 \tag{6}$$

Since $\mathbf{h}_1 = \mathbf{r}_{\mathbf{x}_0 d_0}/\|\mathbf{r}_{\mathbf{x}_0 d_0}\|^2$ is contained in the true signal subspace, it follows that $\mathbf{V_n^H}\ \mathbf{r}_{\mathbf{x}_0 d_0} = 0$. Considering $\mathbf{V_s}\mathbf{V_s}^H + \mathbf{V_n}\mathbf{V_n}^H = \mathbf{I}_{M\times M}$ and (4), (5) can be formulated as

$$\begin{aligned}\mathbf{T}_D &= [\mathbf{V_s}\mathbf{V_s}^H\mathbf{r}_{\mathbf{x}_0 d_0}, \mathbf{V_s}\Lambda_\mathbf{s}\mathbf{V_s}^H\mathbf{r}_{\mathbf{x}_0 d_0}, \ldots, \mathbf{V_s}\Lambda_\mathbf{s}^{(D-1)}\mathbf{V_s}^H\mathbf{r}_{\mathbf{x}_0 d_0}]\mathbf{K}\\ &= \mathbf{V_s}[\mathbf{V_s}^H\mathbf{r}_{\mathbf{x}_0 d_0}, \Lambda_\mathbf{s}\mathbf{V_s}^H\mathbf{r}_{\mathbf{x}_0 d_0}, \ldots, \Lambda_\mathbf{s}^{(D-1)}\mathbf{V_s}^H\mathbf{r}_{\mathbf{x}_0 d_0}]\mathbf{K}\\ &= \mathbf{AT}[\mathbf{V_s}^H\mathbf{r}_{\mathbf{x}_0 d_0}, \Lambda_\mathbf{s}\mathbf{V_s}^H\mathbf{r}_{\mathbf{x}_0 d_0}, \ldots, \Lambda_\mathbf{s}^{(D-1)}\mathbf{V_s}^H\mathbf{r}_{\mathbf{x}_0 d_0}]\mathbf{K}\\ &= \mathbf{AF}\end{aligned} \tag{7}$$

where $\mathbf{F} = \mathbf{T}[\mathbf{V_s}^H\ \mathbf{r}_{\mathbf{x}_0 d_0}, \Lambda_\mathbf{s}\mathbf{V_s}^H\ \mathbf{r}_{\mathbf{x}_0 d_0}, \ldots; \Lambda_\mathbf{s}^{(D-1)}\ \mathbf{V_s}^H\ \mathbf{r}_{\mathbf{x}_0 d_0}]\ \mathbf{K} \in C^{P\times D}$. This shows that $\mathbf{T}_D$ also spans the signal subspace. Therefore, the relation in (7) forms a new basis for signal subspace fitting. Thus a novel criterion function is given by

$$\{\hat{\theta}, \hat{\mathbf{F}}\} = \arg\min_{\theta\mathbf{F}} \|\hat{\mathbf{T}}_D - \mathbf{A}(\theta)\mathbf{F}\|_F^2 \tag{8}$$

where $\hat{\mathbf{T}}\mathrm{z}_D$ is the estimate of $\mathbf{T}_D$. For the fixed unknown parameter $\mathbf{A}(\theta)$, the solution for the linear parameter $\mathbf{F}$ is $\hat{\mathbf{F}} = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{T}_{\hat{D}}$. Substituting it to (8) yields the SSF criterion function:

$$\begin{aligned}\hat{\theta} &= \arg\min_\theta \|\mathbf{P}_\mathbf{A}^\perp\hat{\mathbf{T}}_D\|_F^2\\ &= \arg\min_\theta\{tr[\mathbf{P}_\mathbf{A}^\perp\hat{\mathbf{T}}_D\hat{\mathbf{T}}_D^H]\}\end{aligned} \tag{9}$$

The novel SSF method is different from the WSF based on the eigen-decomposition though they are very similar formally. It can be proved that the columns of $\mathbf{T}_D$ can span a compressed signal subspace the dimension of which may be much less than the number of signals. Hence accurate knowledge of the number of signals is unnecessary in the process of estimating the constringent signal subspace.

Computational cost consideration: It is noticeable that the MSWF algorithm above avoids the formation of blocking matrices, and all the operations of the MSWF merely involve complex vector-vector products, thereby implying computational complexity $O(M)$ per snapshot for each matched filter. Thus, to estimate the signal subspace of rank D, the computational burden of the proposed method is only of $O(D\ M\ N)$ flops in the training model. However, the WSF technique resorts to the estimation of the covariance matrix and its eigen-decomposition, which requires $O(M^2N + M^3)$ flops.

Numerical example 1: The array herein is assumed to be a ULA with 32 isotropic sensors, the spacings of which equal half-wavelength. Suppose that there are three signals impinging upon the array from the same signal source. The first is a direct-path signal and the others are the scaled and delayed replicas of the first signal that represent the multipaths or the 'smart' jammers. The propagation constants are $\{1, -0.8 + j0.6, -0.4 + j0.7\}$. The true DOAs are assumed to be $\{-4^0, 0^0, 5^0\}$. The number of snapshots is 64. The root-mean-squared errors (RMSEs) of signals against signal-to-noise ratio (SNR) for the rank of the MSWF equal to 1 are shown in Fig. 1. It is easily seen that the proposed method outperforms the WSF estimator when SNR is less than −5 dB. When SNR increases, the RMSEs of the two methods approach to the Cramér-Rao bound (CRB).

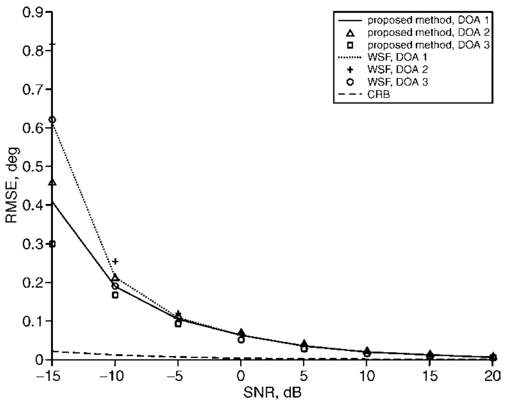

Fig. 1 *RMSEs of signals against SNR for fixed D = 1*

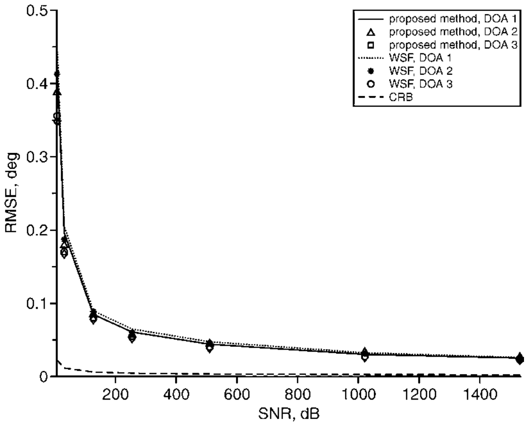

Fig. 2 *RMSEs of signals against number of snapshots for fixed D = 1*

Numerical example 2: The number of sensors is assumed to be 16 and the signals are the same as those of example 1. For the fixed SNR equal to 15 dB, Fig. 2 illustrates that the RMSEs of the proposed method coincide with those of the WSF as the number of snapshots increases, therefore indicating that the proposed technique yields comparable resolution and precision with the WSF method in the case of coherent signals.

Conclusion: The proposed technique does not compute the covariance matrix or its eigenvectors, and does not require the backward recursion of the MSWF. Thus, its computation load is significantly reduced. Moreover, it still can provide comparable performances with the WSF when the dimension of the compressed signal subspace is less than the number of signals.

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